Reduced dynamics of Ward solitons

Maciej Dunajski[*]

and

Nicholas S. Manton[†]

Department of Applied Mathematics and Theoretical Physics

University of Cambridge

Wilberforce Road, Cambridge CB3 0WA, UK

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Abstract

The moduli space of static finite energy solutions to Ward's integrable chiral model is the space M_N of based rational maps from $\mathbb{CP}^1$ to itself with degree N. The Lagrangian of Ward's model gives rise to a Kähler metric and a magnetic vector potential on this space. However, the magnetic field strength vanishes, and the approximate non–relativistic solutions to Ward's model correspond to a geodesic motion on M_N. These solutions can be compared with exact solutions which describe non–scattering or scattering solitons.

[*]email M.Dunajski@damtp.cam.ac.uk
[†]email N.S.Manton@damtp.cam.ac.uk

1 Introduction

There aren't any known examples of Lorentz invariant equations admitting soliton solutions in $2+1$ dimensions. The first order Yang–Mills–Higgs system proposed by Ward [10] almost does the job: the unknowns (A, Φ) are a one–form and a function which depend on local coordinates $x^\mu = (t, x, y)$, and take values in $\mathbf{su}(2)$, the Lie algebra of $SU(2)$. The metric on $\mathbb{R}^{2+1}$ is $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2$. The equations are

$$D\Phi = *F. \tag{1.1}$$

Here $F = (1/2)F_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ is the curvature of a connection $A = A_\mu \mathrm{d}x^\mu$. The action of the covariant derivative D_μ on Φ is $D_\mu\Phi = \partial_\mu\Phi + [A_\mu, \Phi]$, and

$$F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

Solutions of (1.1) are defined modulo the gauge transformations

$$A_\mu \longrightarrow \hat{A}_\mu = gA_\mu g^{-1} - (\partial_\mu g)g^{-1}, \qquad F_{\mu\nu} \longrightarrow \hat{F}_{\mu\nu} = gF_{\mu\nu}g^{-1}, \qquad \Phi \longrightarrow \hat{\Phi} = g\Phi g^{-1},$$

where $g(x^\mu) \in SU(2)$.

This system is integrable in more than one way: it arises as a reduction of $2+2$ dimensional self–dual Yang–Mills equations by a non–null translation, it possesses an infinite number of conserved currents and so forth. It can not however be regarded as a genuine soliton system, because the energy functional associated to the Lagrangian

$$\int \left\{ \frac{1}{2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) - \mathrm{Tr}(D_\mu\Phi D^\mu\Phi) \right\}\mathrm{d}x\mathrm{d}y$$

is not positive definite and its density vanishes on all solutions to (1.1) (the given Lagrangian differs from the standard Yang–Mills–Higgs Lagrangian by the relative sign between the two terms; the second order Euler–Lagrange equations are satisfied by solutions to the first order system (1.1)).

There exists a positive functional associated to (1.1). To see it, note that the equations (1.1) arise as the integrability conditions for an overdetermined system of linear equations

$$(D_y + D_t - \lambda(D_x + \Phi))\psi = 0, \qquad (D_x - \Phi - \lambda(D_t - D_y))\psi = 0, \tag{1.2}$$

where ψ is an $SU(2)$–valued function of (t, x, y) and a complex parameter λ. The integrability conditions for (1.2) imply the existence of a gauge $A_t = A_y$ and $A_x = -\Phi$, and a matrix $J : \mathbb{R}^{2+1} \longrightarrow SU(2)$ such that

$$A_t = A_y = \frac{1}{2}J^{-1}(J_t + J_y), \qquad A_x = -\Phi = \frac{1}{2}J^{-1}J_x,$$

and equations (1.1) become

$$(\eta^{\mu\nu} + V_\alpha \varepsilon^{\alpha\mu\nu})\partial_\mu(J^{-1}\partial_\nu J) = 0. \tag{1.3}$$

Here $\eta^{\mu\nu} = \text{diag}(-1, 1, 1)$ is the inverse of the Minkowski metric, $\varepsilon^{\alpha\mu\nu}$ is the alternating tensor with $\varepsilon^{012} = 1$, and $V_\alpha = (0, 1, 0)$. The energy functional associated with (1.3) is

$$E = \int \frac{1}{2}\delta^{\mu\nu}\text{Tr}(\partial_\mu J \partial_\nu J^{-1})\mathrm{d}x\mathrm{d}y = \int \mathcal{E}\mathrm{d}x\mathrm{d}y, \tag{1.4}$$

and it is conserved. This functional is positive definite, but it came at the price of losing the Lorentz invariance: any choice of a constant space–like vector V_α fixes a direction in $\mathbb{R}^{2+1}$. Finiteness of E is ensured by imposing the boundary condition (valid for all t)

$$J = J_0 + J_1(\theta)r^{-1} + O(r^{-2}) \qquad \text{as} \qquad r \longrightarrow \infty, \qquad x + iy = re^{i\theta}. \tag{1.5}$$

The integrability of equations (1.1), or equivalently of (1.3), allows a construction of explicit static and also time–dependent solutions by twistor or inverse–scattering methods [10, 11]. Static solutions are described by rational maps and may be identified with lumps or sigma model solitons. There are time–dependent solutions with non–scattering solitons [10], and also solitons that scatter [12].

In this paper we choose a different route and seek slow–moving solitons using a modification of the geodesic approximation [5] which may involve a background magnetic field in the moduli space of static solutions. The argument is based on the analogy with a particle in $\mathbb{R}^n$ moving in a potential U and coupled to a magnetic vector potential $\mathbf{A}(\mathbf{q})$; the Lagrangian is

$$L = \frac{1}{2}|\dot{\mathbf{q}}|^2 + \mathbf{A}\cdot\dot{\mathbf{q}} - U(\mathbf{q}),$$

where $U : \mathbb{R}^n \to \mathbb{R}$ is a potential whose minimum value is 0. The equilibrium positions are on a subspace $M \subset \mathbb{R}^n$ given by $U = 0$. If the kinetic energy of the particle is small, and the initial velocity is tangent to M, the exact motion will be approximated by a motion on M with the Lagrangian L' given by a restriction of L to M

$$L' = \frac{1}{2}h_{jk}\dot{\gamma}_j\dot{\gamma}_k + A_j\dot{\gamma}_j. \tag{1.6}$$

Here, the γs are local coordinates on M, and the metric h and the one–form $A_j\mathrm{d}\gamma_j$ are induced on M from the Euclidean inner product and the magnetic vector potential $\mathbf{A}$ respectively. If for example $U = (1 - r^2)^2$, where $r = |\mathbf{q}|$, and the magnetic field is constant, then the motion with small energy is approximated by the motion on the unit sphere in $\mathbb{R}^n$ where trajectories are small circles, that is, a circular motion at $r = 1$ with constant speed.

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In the absence of the magnetic term we expect the true motion to have small oscillations in the direction transverse to M, with the approximation becoming exact at the limit of zero initial velocity. The presence of a magnetic force may in some cases balance the contribution from a centrifugal force so that the oscillations do not occur, and the exact motion is confined to M.

The dynamics of finite energy solutions to (1.3) will be put in this framework with $\mathbb{R}^n$ replaced by an infinite–dimensional configuration space of the field J, and M replaced by M_N (the moduli space of rational maps from $\mathbb{CP}^1$ to itself with degree N), the important point being that the static solutions to (1.3) give the absolute minimum of the potential energy for given N. The time–dependent solutions to (1.3) with small total energy (hence small potential energy) above the absolute minimum will be approximated by a sequence of static states, that is a motion in M_N.

This comes down to three steps:

1. Construct finite–dimensional families of static solutions to (1.3) with finite energy.

2. Allow time–dependence of the parameters, and read off the metric h and the magnetic one–form A on the moduli space from the Lagrangian for J. Investigate whether A has a non–vanishing or vanishing magnetic two–form $F = \mathrm{d}A$. Some of the parameters may have to be fixed artificially to ensure that this metric is complete, and all tangent vectors have finite length.

3. The geodesic motion, possibly with magnetic forcing, should then approximate the slow (non–relativistic) motion of rational map, or lump solutions to (1.3).

In the next section we shall find the metric and the one–form to be

$$h_{jk} = 8 \int_{\mathbb{R}^2} \frac{|\partial_j f \partial_k f|}{(1 + |f|^2)^2} \mathrm{d}x\mathrm{d}y, \qquad A_j = 4\pi \int_{\mathbb{R}^2} \frac{\mathrm{Re}(\partial_z f \partial_j \overline{f})}{(1 + |f|^2)^2} \mathrm{d}x\mathrm{d}y. \qquad (1.7)$$

Here $f = f(z)$ is a rational meromorphic function of z, which depends on real parameters γ_j, and $\partial_j f = \partial f / \partial \gamma_j$. We shall show that the magnetic two–form $F = \mathrm{d}A$ in fact vanishes on the moduli space[1].

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[1]It is worth remarking that even magnetically forced geodesic motion can be regarded as a geodesic motion on an S^1–bundle $\mathcal{L} \to M_N$ equipped with a connection ω with curvature F. In a local trivialisation $\omega = \mathrm{d}\theta + A$, where $\theta \in S^1$ is a coordinate along the fibres. The motion in a magnetic field on M_N given by (1.6) is geodesic on $\mathcal{L}$ with respect to a Kaluza–Klein metric $\hat{h} = h + \omega \otimes \omega$. This can be verified by writing the Euler–Lagrange equations of

$$\hat{L} = \frac{1}{2}h(\dot{\gamma}, \dot{\gamma}) + \frac{1}{2}(\dot{\theta} + h(A, \dot{\gamma}))^2.$$

2 Reduced dynamics

All static solutions to (1.3) are just chiral fields on $\mathbb{R}^2$, i.e. solutions to

$$\overline{\partial_z}(J^{-1}\partial_z J) + \partial_z(J^{-1}\overline{\partial_z}J) = 0 \tag{2.1}$$

where $z = x + iy$, and $\partial_z = \partial/\partial z$.

It is convenient at this point to fix J_0 to lie in the equator $S^2 \subset SU(2)$ given by $J_0^2 = -\mathbf{1}$. We shall choose $J_0 = i\sigma^1$, where $\sigma^1, \sigma^2, \sigma^3$ are the Pauli matrices. It has been shown in [8] that all chiral fields with finite energy and satisfying our boundary conditions can be globally conjugated into the so–called 1–uniton solutions

$$J = \frac{i}{1+|f|^2}\begin{pmatrix} 1 - |f|^2 & 2f \\ 2\overline{f} & |f|^2 - 1 \end{pmatrix}, \tag{2.2}$$

where the holomorphic function f is rational in z and $f(z) \longrightarrow 1$ as $|z| \longrightarrow \infty$. General solutions to the chiral equations are $SU(2) = S^3$ valued, but the 1–uniton (2.2) everywhere takes values in the equator $J^2 = -\mathbf{1}$, and is in effect a based harmonic map from a two–sphere (conformal compactification of $\mathbb{R}^2$) to itself. All such maps are classified by integer winding numbers N with values in $\pi_2(S^2) = \mathbb{Z}$. This integer is precisely the degree of f: for a given N, f is of the form

$$f(z) = \frac{p(z)}{q(z)} = \frac{(z - q^1)...(z - q^N)}{(z - q^{N+1})...(z - q^{2N})}, \tag{2.3}$$

and the map f is an N–fold covering of the target $\mathbb{CP}^1 = S^2$.

Let $M_N \subset \mathbb{C}^{2N}$ be the moduli space of 1–unitons of degree N. This space consists of all based rational functions of degree N (we assume that the denominator and the numerator in (2.3) have no common factors) and has real dimension $4N$. Let the parameters of the solution be denoted collectively by $\gamma = (q, \overline{q})$, and let $J(\gamma)$ be the corresponding solution to (2.1) (this solution also depends on x, y). Let $\gamma(t)$ be a path in M_N. Then $J(\gamma(t))$ is not in general a solution to the time–dependent eq.(1.3), but for $t = 0$ it provides initial data for J and its derivative. The initial velocity

$$\dot{J}|_{t=0} = \frac{\partial J(\gamma(t))}{\partial \gamma_i}\dot{\gamma}_i\Big|_{t=0}$$

is tangent to M_N, and is a linearised solution to (2.1). If this initial velocity is small, the true dynamical motion will remain close to M_N because of the conservation of energy. In the

One equation implies that $\dot{\theta} + h(A, \dot{\gamma}) = C$ is a constant which can be chosen so that the remaining equations coincide with those obtained from (1.6).

limiting case, when the velocity remains small, the motion will be governed by a Lagrangian of the form (1.6).

To find this moduli space Lagrangian we need the action for J, which is a sum of a standard part quadratic in the derivatives of J, and a Wess–Zumino–Witten (WZW) like term [14, 15]. This involves an extended field

$$\hat{J} : \mathbb{R}^{2+1} \times [0, 1] \longrightarrow SU(2)$$

such that $\hat{J}(x^\mu, 0)$ is a constant group element, which we take to be the identity element, and $\hat{J}(x^\mu, 1) = J(x^\mu)$. That is to say $\hat{J}$ is defined in the interior of a cylinder which has the space–time as one of its boundary components. The action is

$$S = S_C + S_M, \tag{2.4}$$

where

$$S_C = -\int_{\mathbb{R}^2} \int_{t_1}^{t_2} \frac{1}{2} \mathrm{Tr}((J^{-1}J_t)^2 - (J^{-1}J_x)^2 - (J^{-1}J_y)^2) \mathrm{d}t\mathrm{d}x\mathrm{d}y,$$

$$S_M = -\int_{\mathbb{R}^2} \int_{t_1}^{t_2} \int_0^1 \frac{1}{3} V_p \varepsilon^{pqrs} \mathrm{Tr}(\hat{J}^{-1}\partial_q \hat{J} \hat{J}^{-1}\partial_r \hat{J} \hat{J}^{-1}\partial_s \hat{J}) \mathrm{d}\rho\mathrm{d}t\mathrm{d}x\mathrm{d}y.$$

The indices p, q, r, s take values $0, 1, 2, 3$, where $x^3 = \rho$ and $V = (0, 1, 0, 0)$. In [3] it was demonstrated that the variation of the action does not depend on the choice of the extension $\hat{J}$, and leads to the Ward model equation (1.3).

The kinetic part of the effective Lagrangian (1.6) can be found as follows: Given a path $\gamma(t)$ in M_N we define the kinetic energy at time t by

$$T[J] = -\frac{1}{2} \int \mathrm{Tr}(J^{-1}J_t)^2 \mathrm{d}x\mathrm{d}y.$$

Substituting for J from (2.2) yields

$$h(\dot{\gamma}(t), \dot{\gamma}(t)) = 2T[J] = \int \frac{8|\dot{f}|^2}{(1 + |f|^2)^2} \mathrm{d}x\mathrm{d}y, \tag{2.5}$$

where f depends on $\gamma(t)$, and hence $\dot{f}$ depends on γ and $\dot{\gamma}$. Expressing $\dot{f}$ as $\partial_j f \dot{\gamma}_j$ verifies the first part of (1.7).

Notice that there is no potential term in the Lagrangian (1.6), since for the static solutions (2.2), the potential energy part of S_C is minimised by the degree of the rational function f:

$$E = -\frac{1}{2} \int \mathrm{Tr}((J^{-1}J_x)^2 + (J^{-1}J_y)^2) \mathrm{d}x\mathrm{d}y = 4i \int_{S^2} \frac{\mathrm{d}f \wedge \mathrm{d}\overline{f}}{(1 + |f|^2)^2} = 8N\pi, \tag{2.6}$$

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where in the last integral we have used the fact that the solution (2.2) extends to a one–point compactification of $\mathbb{R}^2$, and that the rational function (2.3) is an N–fold covering of the two–sphere. This constant potential energy can be dropped.

The analysis of slowly moving lumps in the $\mathbb{CP}^1$ model leads to an identical expression for the kinetic energy [9]. (This was to be expected, because the conserved functional (1.4) is identical to that of the non–integrable chiral equation obtained by setting $V_\alpha = 0$.) The slow dynamics could however be different for these two models: trajectories of slow moving $\mathbb{CP}^1$ lumps are just the geodesics of h, but the motion of lumps of (1.3) would be affected by any magnetic field F on the moduli space.

Using the cyclic property of the trace we can simplify the integrand of the WZW term S_M to

$$\mathcal{L}_M = \mathrm{Tr}([\hat{J}^{-1}\hat{J}_y, \hat{J}^{-1}\hat{J}_t]\hat{J}^{-1}\hat{J}_\rho).$$

One can now understand the vanishing of F. The variation of S_M involves the integral over $\mathbb{R}^2 \times [t_1, t_2]$ of $\mathrm{Tr}(J^{-1}\delta J[J^{-1}J_t, J^{-1}J_y])$. If J is restricted to the equator $J^2 = -\mathbf{1}$, for all x, y and t, then $J^{-1}\delta J$, $J^{-1}J_t$ and $J^{-1}J_y$ lie in a two–dimensional subspace of $\mathbf{su}(2)$ at any given space–time point, and the trace above vanishes. In the moduli space approximation, J is restricted in this way, and the variation of the action under a change of path in the moduli space (with fixed endpoints) has no contribution from the WZW term. There is therefore no magnetic force in the reduced equation of motion.

Despite this, we shall calculate the 'magnetic' one–form A in (1.6) from the WZW term. We make the ansatz

$$\hat{J}(x^\mu, \rho) = \cos g(\rho)\mathbf{1} + \sin g(\rho)J,$$

where J is the static solution given by (2.2), and $g(\rho)$ is any smooth function on the interval $[0, 1]$ such that $g(0) = 0, g(1) = \pi/2$. This, with the help of $J^2 = -\mathbf{1}, J^* = -J$ and some algebra, reduces $\mathcal{L}_M$ to

$$\mathcal{L}_M = \sin^2 g(\rho)\frac{\mathrm{d}g(\rho)}{\mathrm{d}\rho}\mathrm{Tr}(J[J_y, J_t]),$$

and the magnetic one–form on the moduli space can be read off from this Lagrangian density

$$\int_{t_1}^{t_2} A_i(\gamma)\dot{\gamma}^i\mathrm{d}t = \int_{t_1}^{t_2}\int_{\mathbb{R}^2\times[0,1]}\mathcal{L}_M\mathrm{d}x\mathrm{d}y\mathrm{d}\rho\mathrm{d}t.$$

Using

$$\mathrm{Tr}\left(J\left[\frac{\partial J}{\partial f}, \frac{\partial J}{\partial \overline{f}}\right]\right) = -\frac{8i}{(1 + |f|^2)^2}$$

we find

$$A_i(\gamma)\dot{\gamma}^i = \frac{\pi}{4}\int \mathrm{Tr}(J[J_y, J_t])\mathrm{d}x\mathrm{d}y = 4\pi\int \frac{\mathrm{Re}(f_z\dot{\overline{f}})}{(1 + |f|^2)^2}\mathrm{d}x\mathrm{d}y, \tag{2.7}$$

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which establishes (1.7).

2.1 Kähler and magnetic potentials

The metric (2.5) is known to be Kähler [6, 7], with the holomorphic coordinates defined to be any functions of $(q^\alpha) = (q^1, ..., q^{2N})$. Let $d = \partial + \bar{\partial}$ be a decomposition of the total derivative on this Kähler manifold, and let $f = p(z)/q(z)$ be given by (2.3). Then

$$A = \partial\left(\int \chi dx dy\right) + \bar{\partial}\left(\int \bar{\chi} dx dy\right), \qquad \text{where} \qquad \chi = 2\pi\frac{\partial}{\partial \bar{z}}\left(\ln\left(1 + |f|^2\right)\right),$$

so both the magnetic field $F = dA$, and the metric h, can be written in terms of scalar potentials

$$F = i\partial \wedge \bar{\partial}\Omega_F \in \Lambda^{(1,1)}(M_N), \qquad h = \frac{\partial^2\Omega_h}{\partial q^\alpha \partial q^{\bar{\beta}}}\, dq^\alpha dq^{\bar{\beta}},$$

where

$$\Omega_F = -2\pi\int_{\mathbb{R}^2} \frac{\partial}{\partial y}\ln\left(|p|^2 + |q|^2\right)dx dy, \qquad \Omega_h = 8\int_{\mathbb{R}^2}\ln\left(|p|^2 + |q|^2\right)dx dy, \qquad (2.8)$$

and we have used the freedom of adding any holomorphic or antiholomorphic functions of q^α to the potentials Ω_F and Ω_h.

The equations of motion in the moduli space approximation in holomorphic coordinates are

$$\ddot{q}^\alpha + \Gamma^\alpha_{\beta\gamma}\dot{q}^\beta\dot{q}^\gamma = -h^{\alpha\bar{\beta}}F_{\gamma\bar{\beta}}\dot{q}^\gamma$$

(and the complex conjugate of this), where

$$F_{\alpha\bar{\beta}} = 2i\frac{\partial^2\Omega_F}{\partial q^\alpha \partial q^{\bar{\beta}}}, \qquad h_{\alpha\bar{\beta}} = \frac{\partial^2\Omega_h}{\partial q^\alpha \partial q^{\bar{\beta}}}, \qquad \Gamma^\alpha_{\beta\gamma} = h^{\alpha\bar{\delta}}\frac{\partial h_{\beta\bar{\delta}}}{\partial q^\gamma},$$

and formulae (2.8) imply that $f(z)$ and $f(z)^{-1}$ give rise to the same dynamics on moduli space.

However we shall now show that a suitable regularisation of the magnetic scalar potential leads to the vanishing of F. Set

$$p = z^n + az^{n-1} + ..., \qquad q = z^n + bz^{n-1} + ..., \qquad (2.9)$$

and consider Ω_F in (2.8) with the integral over $\mathbb{R}^2$ replaced by the integral over an annulus $D(\epsilon, R) = \{z = r\exp(i\theta), \epsilon < r < R\}$ bounded by circles C_R and C_ϵ of radii R and ϵ respectively. We will regularize the integrand by subtracting $\partial/\partial y(\ln 2|z|^{2n})$, as this term does not contribute to F. The application of Green's theorem gives

$$\Omega_F = \lim_{R\to\infty}\lim_{\epsilon\to 0}\int_{D(\epsilon,R)} 2\pi\frac{\partial}{\partial y}\ln\left(\frac{|p|^2 + |q|^2}{2|z|^{2n}}\right)dx dy$$

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$$= \lim_{R \to \infty} \lim_{\epsilon \to 0} \left(\oint_{C_R} - \oint_{C_\epsilon} \right) 2\pi \ln \left(\frac{|p|^2 + |q|^2}{2|z|^{2n}} \right) \mathrm{d}x$$

$$= \lim_{R \to \infty} \int_0^{2\pi} 2\pi R \ln \left(1 + \frac{c_1}{R} + \frac{c_2}{R^2} + ... \right) \sin\theta \mathrm{d}\theta$$

$$- \lim_{\epsilon \to 0} \int_0^{2\pi} 2\pi\epsilon (\ln \left(|p|^2 + |q|^2 \right) - 2n \ln \epsilon - \ln 2) \sin\theta \mathrm{d}\theta$$

where c_a depend on θ and

$$c_1 = \frac{1}{2}(a + b) \exp(-i\theta) + \frac{1}{2}(\overline{a} + \overline{b}) \exp i\theta.$$

The second limit vanishes and the first term can be expanded for large R to give

$$\Omega_F = \lim_{R \to \infty} \left(2\pi \int_0^{2\pi} c_1 \sin\theta \mathrm{d}\theta + O\left(\frac{1}{R} \right) \right) = 2\pi^2 \operatorname{Im} (a + b), \tag{2.10}$$

which is a sum of holomorphic and antiholomorphic functions on M_N. Therefore

$$F = 0.$$

We can give a deeper geometrical interpretation of the metric and magnetic field on M_N. The static solutions take values in the Kähler manifold $S^2 \subset S^3$ and (following the argument of Ruback [6]) the Kähler structure on M_N is induced from $\mathbb{CP}^1$ as follows. Let

$$p = (x, y), \qquad X \in T_f S^2, \qquad X(p) = \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t), p)|_{t=0}.$$

If $(\hat{h}_{f(p)}, \hat{\mathcal{J}}_{f(p)})$ is the standard Kähler structure on $T_{f(p)} S^2$, then

$$h_f(X, X) = \int_{\mathbb{R}^2} \hat{h}_{f(p)}(X(p), X(p)) \mathrm{d}x \mathrm{d}y, \qquad (\mathcal{J}_f(X))(p) = \hat{\mathcal{J}}_f(p)(X(p))$$

give a metric and an almost complex structure on $T_\gamma M_N$. It can be formally shown that $(h, \mathcal{J})$ is in fact a Kähler structure.

The magnetic one–form $A_f \in T^*_\gamma M_N$ given by (1.7) can be similarly constructed in terms of a one–form $\hat{A}$ which is dual to the push–forward of the vector field V (compare (1.3)) from $\mathbb{R}^{2+1}$. Explicitly

$$f_*(V) = f_*(\partial/\partial x) = \frac{\partial f}{\partial z} \frac{\partial}{\partial f} + \frac{\partial \overline{f}}{\partial \overline{z}} \frac{\partial}{\partial \overline{f}}, \qquad \hat{A} = \hat{h}(f_*(V), ...) = 8 \frac{f_z \mathrm{d}\overline{f} + \overline{f}_z \mathrm{d}f}{(1 + |f|^2)^2}$$

and

$$X \lrcorner A_f = \int_{\mathbb{R}^2} (X(p) \lrcorner \hat{A}_{f(p)}) \mathrm{d}x \mathrm{d}y = 8 \int_{\mathbb{R}^2} \dot{\gamma}_j \frac{f_z \overline{\partial_j f} + \overline{f}_z \partial_j f}{(1 + |f|^2)^2} \mathrm{d}x \mathrm{d}y,$$

which coincides with (1.7) up to a constant factor if $X = \dot{\gamma}_j \partial/\partial \gamma_j$.

To understand the appearance of the one–form A in this context, one may also consider a connection on $SU(2)$ (the full target space of J) which holds the left–invariant vectors covariantly constant. This connection is flat, but necessarily has torsion which is totally antisymmetric and therefore gives rise to a closed three–form T. Let B be a locally defined two–form such that $T = \mathrm{d}B$. The magnetic term S_M in the action (2.4) is proportional to the integral of f^*B over the space-time. Its variation vanishes if B is restricted to the equatorial two–sphere in $SU(2)$.

3 Reduced dynamics of the K–equation.

In this section we shall carry out the moduli space approximation in a different potential formulation of (1.1).

Choose the familiar gauge $A_y = A_t, A_x = -\Phi$. The vanishing of the term proportional to λ in the compatibility conditions (1.2) implies the existence of $K : \mathbb{R}^{2+1} \longrightarrow \mathbf{su}(2)$ such that

$$A_y = A_t = \frac{1}{2}K_x, \qquad A_x = -\Phi = \frac{1}{2}(K_t - K_y).$$

The 0th order term in the compatibility conditions now yields

$$K_{tt} - K_{xx} - K_{yy} + [K_x, K_t - K_y] = 0. \tag{3.1}$$

The relation between $K \in \mathbf{su}(2)$ and $J \in SU(2)$ is

$$K_x = J^{-1}(J_t + J_y), \qquad K_t - K_y = J^{-1}J_x,$$

and exhibits a duality between the two formulations: the compatibility condition $K_{xt} - K_{xy} = K_{tx} - K_{yx}$ yields the field equation (1.3).

The K–equation (3.1) admits a Lagrangian formulation with the Lagrangian density

$$-\operatorname{Tr}\left(\frac{1}{2}((K_t)^2 - (K_x)^2 - (K_y)^2) - \frac{1}{3}K[K_x, K_t - K_y]\right). \tag{3.2}$$

For the time–independent solutions we have $J^{-1}J_z = -iK_z$, $J^{-1}J_{\bar{z}} = iK_{\bar{z}}$, and integrating these equations gives, surprisingly,

$$K = J$$

where J is given by (2.2). This makes sense because the unit two–sphere in the Lie algebra $\mathbf{su}(2)$ may be identified with the equatorial two–sphere in $SU(2)$. In general, $J = a_0\mathbf{1} + i\mathbf{a} \cdot \sigma$, where a_0 and $\mathbf{a} \in \mathbb{R}^3$ depend on (t, x, y). For the static solution (2.2), $a_0 = 0$ and $|\mathbf{a}| = 1$ so

$J^2 = -\mathbf{1}, \mathrm{Tr}(J) = 0$, which is the equatorial condition. But this means that $K = J$ lies in the Lie algebra, and moreover $K \in S^2 \subset \mathbb{R}^3 \cong \mathbf{su}(2)$.

The energy densities of static solutions to (1.3) and (3.1) are proportional but not equal as

$$3\,\mathrm{Tr}\Big(\frac{1}{2}(K_x^2 + K_y^2) - \frac{1}{3}K[K_x, K_y]\Big) = \mathrm{Tr}\Big(\frac{1}{2}(J^{-1}J_x)^2 + \frac{1}{2}(J^{-1}J_y)^2\Big).$$

The potential energy in the K–formulation is therefore $8N\pi/3$ and again can be dropped.

Now consider the slow–motion approximation, where K is expressed in terms of a rational holomorphic function f which depends on time through the collection of $4N$ parameters $\gamma(t)$. Then the kinetic energy term $-(1/2)\int \mathrm{Tr}(K_t)^2\mathrm{d}x\mathrm{d}y$ gives rise to the metric (2.5) (because $K_t^2 = (J^{-1}J_t)^2$). We conclude that the Kähler structures on the moduli spaces of static solutions to (1.3) and (3.1) are the same.

The term in (3.2) of first order in K_t implies that there is a magnetic one–form on the space of fields K. This is given by

$$
\begin{aligned}
(A_K)_j(\gamma)\dot{\gamma}^j &= \frac{1}{3}\int \mathrm{Tr}(K[K_x, K_t])\mathrm{d}x\mathrm{d}y = \frac{16}{3}\int \frac{\mathrm{Im}(f_z\dot{\bar{f}})}{(1+|f|^2)^2}\mathrm{d}x\mathrm{d}y \qquad (3.3)\\
&= i\partial\Big(\int \chi\mathrm{d}x\mathrm{d}y\Big) - i\bar{\partial}\Big(\int \bar{\chi}\mathrm{d}x\mathrm{d}y\Big), \quad \text{where} \quad \chi = \frac{8}{3}\frac{\partial}{\partial z}\Big(\ln\big(1+|f|^2\big)\Big),
\end{aligned}
$$

and $F = i\partial \wedge \bar{\partial}\Omega_{F_K}$ where

$$\Omega_{F_K} = -\frac{8}{3}\int_{\mathbb{R}^2} \frac{\partial}{\partial x}\ln\big(|p|^2 + |q|^2\big)\mathrm{d}x\mathrm{d}y.$$

The scalar potentials for the magnetic two–form are therefore different in the J and K formulations. Nevertheless the limiting procedure described in the last section also applies in this case (with y replaced by x) leading to

$$\Omega_{F_K} = -\frac{8}{3}\pi\mathrm{Re}(a + b).$$

As before, the magnetic two–form vanishes when restricted to the moduli space of finite energy static solutions.

4   Examples

In the moduli space approximation J stays on the equatorial $S^2 \subset S^3$, and the lumps are located where J departs from its asymptotic value (1.5). In these regions the energy density of (1.4) attains its local maxima. The velocities of the lumps are the velocities of these local maxima.

The charge one solution is given by

$$f = \alpha + \frac{\beta}{z + \gamma}, \tag{4.1}$$

and we need to fix α and β in order for (2.5) to be well defined. Choosing $\alpha = 0, \beta = 1, \gamma = \gamma(t)$, and setting $\gamma(t) = R(t) \exp(i\theta(t))$ we find the metric and the one–form

$$h = 8\pi(\mathrm{d}R^2 + R^2\mathrm{d}\theta^2), \qquad A = 4\pi^2\mathrm{d}(R\cos\theta).$$

Therefore the metric is flat, and the motion is along straight lines, $\gamma(t) = -vt$, because $\mathrm{d}A = 0$ does not contribute to the Euler–Lagrange equations. The energy density is approximated by

$$\mathcal{E} = (1 + |z - vt|^2)^{-2}.$$

Next we look at the charge two case[2]

$$f = \alpha + \frac{\beta z + \gamma}{z^2 + \delta z + \kappa}. \tag{4.2}$$

The corresponding metric was constructed by Ward [9]. The parameters α, β have to be fixed to ensure finiteness of kinetic energy, and δ can be set to 0 by exploiting the translational invariance of (2.2). Moreover the Möbius transformations can be used to ensure $\alpha = 0, \beta \in \mathbb{R}$, and here Ward makes an additional choice $\beta = 0$. The resulting metric is therefore defined on four–dimensional leaves of a foliation of M_2, with local coordinates $(\gamma, \overline{\gamma}, \kappa, \overline{\kappa})$. The Kähler potential is given by

$$\Omega_h = -4\pi|\kappa| + \pi|\gamma| \int_0^{\pi/2} \sqrt{1 + |\kappa/\gamma|^2 \sin^2\theta} \, \mathrm{d}\theta.$$

The structure is invariant under the torus action and a homothety

$$\gamma \to \exp(i\tau_1)\gamma, \qquad \kappa \to \exp(i\tau_2)\kappa, \qquad |\gamma|^2 + |\kappa|^2 \to \tau_3(|\gamma|^2 + |\kappa|^2).$$

5   Comparision with exact solutions

One method [10] of constructing explicit solutions is based on the associated linear problem (1.2). Let $\psi(x^\mu, \lambda)$ be the fundamental solution to the Lax pair (1.2) (think of ψ as a 2×2 matrix), and let $u = (t + y)/2, v = (t - y)/2$. Then

$$\begin{aligned} A_u - \lambda(A_x + \Phi) &= [-\partial_u\psi + \lambda\partial_x\psi]\psi^{-1} \\ A_x - \Phi - \lambda A_v &= [-\partial_x\psi + \lambda\partial_v\psi]\psi^{-1}, \end{aligned} \tag{5.1}$$

---

[2]We remark that the boundary condition for f in these examples is $f(z) \longrightarrow 0$ as $|z| \longrightarrow \infty$, and hence $J_0 = i\sigma^3$. The conclusion $F = 0$ does not depend on these boundary conditions.

and in the gauge leading to (1.3), $J(x, y, t) = \psi(x, y, t, 0)^{-1}$. This can be an effective method of finding solutions (also known as the 'Riemann problem with zeros'), if we know $\psi(x^\mu, \lambda)$ in the first place. One class of solutions can be obtained by assuming that

$$\psi = \mathbf{1} + \sum_{k=1}^{n} \frac{M_k(x, y, t)}{\lambda - \mu_k}, \qquad \mu_k = \text{const.}$$

The unitarity condition $\psi(x^\mu, \lambda)\psi(x^\mu, \overline{\lambda})^* = \mathbf{1}$ implies rk $M_k = 1$, and demanding that the RHS of (5.1) is linear in λ (like the LHS) yields $M_k = M_k(\omega_k)$, where

$$\omega_k = u\mu_k^2 + x\mu_k + v.$$

Finally (see [10] for details)

$$(J^{-1})_{\alpha\beta} = \chi^{-1/2}(\delta_{\alpha\beta} + \sum_{k,l} \mu_k^{-1}(\Gamma^{-1})^{kl}\overline{m}_\alpha{}^l m_\beta^k). \tag{5.2}$$

Here

$$\Gamma^{kl} = \sum_{\alpha=1}^{2}(\overline{\mu}_k - \mu_l)^{-1}\overline{m}_\alpha{}^k m_\alpha{}^l, \qquad \chi = \prod_{k=1}^{n} \frac{\overline{\mu}_k}{\mu_k},$$

and $m_\alpha{}^k = (1, f_k)$.

The soliton solutions correspond to rational functions $f_k(\omega_k)$. To recover the static solution (2.2) put $n = 1, \mu = i$. The static N lumps are positioned at $(q^1, ..., q^N)$, as the maxima of $\mathcal{E}$ occur at these points. For $\mu \neq \pm i$ there is time dependence, and $n > 1$ corresponds to n solitons moving with different velocities which however do not scatter.

The solution (5.2) with $n = 1$ and $\mu_1 = m\exp i\theta$ is given by

$$J_1 = \frac{1}{1 + |f|^2}\begin{pmatrix} e^{i\theta} + e^{-i\theta}|f|^2 & 2i\sin\theta\, f \\ 2i\sin\theta\, \overline{f} & e^{-i\theta} + e^{i\theta}|f|^2 \end{pmatrix}, \tag{5.3}$$

where $f = f(u\mu^2 + x\mu + v)$ is a holomorphic, rational function. The energy density

$$\mathcal{E} = 2\sin^2\theta\frac{(1 + m^2)^2|f'|^2}{m^2(1 + |f|^2)^2}$$

has local maxima which give the locations $\{(x_a, y_a), a = 1, ..., N\}$ of N lumps. The velocities $(\dot{x}_a, \dot{y}_a) = (-2m\cos\theta/(1 + m^2), (1 - m^2)/(1 + m^2))$ are the same for each lump so (5.3) should be regarded as a one–soliton solution. To make contact with the moduli space approximation write $J_1 = \cos\theta\mathbf{1} + i\mathbf{a} \cdot \sigma$ to reveal that $\cos\theta$ measures the deviation of J_1 from the unit sphere in the Lie algebra $\mathbf{su}(2)$. If J is initially tangent to the space of static solutions, then $\cos\theta = 0$, and we can set $\mu = i(1 + \varepsilon)$, where $\varepsilon \in \mathbb{R}$. The solution is of the form (2.2), but f is rational in

$$\omega = z + \varepsilon(z + it) + \frac{\varepsilon^2}{2}\left(\frac{z - \overline{z}}{2} + it\right),$$

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so

$$f(\omega) = \frac{(z - Q_1)...(z - Q_N)}{(z - Q_{N+1})...(z - Q_{2N})},$$

where the Qs are linear functions of $(\varepsilon^2 \bar{z}, \varepsilon t)$. The (squared) velocity is

$$\mathcal{V}^2 = 1 - 4(1 + \varepsilon)^2/(1 + (1 + \varepsilon)^2)^2,$$

so in the non–relativistic limit (which underlies the moduli space approximation) we regard ε as small. Therefore the Qs depend only on t, and they all move at velocity ε. Setting $N = 1$, we recover the charge one solution (4.1). More generally we find that J is given by (2.2) with

$$f = f_2(z) + t f_1(z),$$

where $f_2 = f(\omega)|_{\varepsilon=0}$ and $f_1 = \partial f/\partial \varepsilon|_{\varepsilon=0}$ are rational functions of z.

Allowing ψ to have poles of order higher than one gives solutions which exhibit soliton scattering. Explicit time–dependent solutions corresponding to scattering can be obtained by choosing $\mu_1 = i + \varepsilon$, $\mu_2 = i - \varepsilon$, and taking the limit $\varepsilon \to 0$. This yields [12]

$$J_2 = \left(1 - \frac{2p_1{}^* \otimes p_1}{||p_1||^2}\right)\left(1 - \frac{2p_2{}^* \otimes p_2}{||p_2||^2}\right), \tag{5.4}$$

where

$$p_1 = \left(1, -\frac{i}{2}f_1\right), \qquad p_2 = (1 + \frac{1}{4}|f_1|^2)\left(1, -\frac{i}{2}f_1\right) - if\left(\frac{i}{2}\overline{f_1}, -1\right), \qquad f(z,t) = f_2 + t f_1',$$

and f_1 and f_2 are rational functions of z. In [12] the 90 degree scattering was illustrated by choosing $f_1 = 2iz, f_2 = 2iz^2$. More complicated examples were considered in [2, 1].

It was recently observed [4] that the total (kinetic+gradient) energy of the solution (5.4) is quantised, and equal to $8\pi N$, where generically $N = 2 \deg f_1 + \deg f_2$. However, $N = \max(2 \deg f_1, \deg f_2)$ if both f_1, f_2 are polynomials. Therefore for all t the total energy of (5.4) is equal to the energy (2.6) of some static solution (2.2).

Solutions to (1.3) obtained in the moduli space approximation have energies close to their potential energy (2.6) as their kinetic energy is small. We should therefore expect that some of these approximate solutions arise from (5.4) by a limiting procedure.

To demonstrate how this limiting procedure is achieved first observe that solutions to (1.3) are defined up to a multiple by a constant element of $SU(2)$. The static solution (2.2) with $f = f_2$ arises from (5.4) by using this freedom and setting $f_1 = 0$

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} J_2|_{f_1=0} = \frac{i}{1 + |f_2|^2}\begin{pmatrix} 1 - |f_2|^2 & 2f_2 \\ 2\overline{f_2} & |f_2|^2 - 1 \end{pmatrix} = J_{\text{static}}.$$

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Moreover the energy density of (5.4) has maxima where $f = f_2 + tf'_1 = 0$. The lumps are located at the zeros $z_a = z_a(t)$, $a = 1, ..., \deg f$ of f and the squared velocity of each lump is

$$\mathcal{V}_a^2 = \frac{|f'_1|^2}{|f'_2 + tf''_1|^2}\bigg|_{z=z_a},$$

so that $|f'_1|^2$ is small in the non–relativistic limit. Therefore $|f_1|$ is also small as we choose J_2 to be tangent to the space of static solutions at $t = 0$. Keeping only the linear terms in f_1 in (5.4) yields

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} J_2 = \frac{i}{1+|f|^2} \begin{pmatrix} 1 - |f|^2 - i(f_2\overline{f_1} + \overline{f_2}f_1) & 2f \\ 2\overline{f} & |f|^2 - 1 - i(f_2\overline{f_1} + \overline{f_2}f_1) \end{pmatrix}.$$

The term $(f_2\overline{f_1} + \overline{f_2}f_1)$ can also be dropped by rescaling the coordinates $x^\mu \to x^\mu/\varepsilon$.

Comparing the resulting expression with (2.2) will give a motion on the moduli space of static solutions if f_2, tf'_1 and f_1^2 lie in the common space of rational maps of degree $\deg f_2$. To achieve this, we therefore take

$$f_1 = \frac{p(z)}{q(z)}, \qquad f_2 = \frac{r(z)}{q(z)^2},$$

where r is of degree $2n$ and p and q are of degree at most n. This is one of the non–generic cases in the analysis of [4], and the total energy is equal to $8\pi \deg f_2$. The resulting motion on the moduli space of static solutions of charge $\deg f_2$ is given by (2.2) with

$$f(z,t) = \frac{r + t(p'q - pq')}{q^2}. \tag{5.5}$$

This motion is restricted to a geodesic submanifold as the parameters in the denominator of f are fixed. In particular, setting $q = 1$, we can take $f_2(z)$ to be a polynomial of degree $2n$ and $f_1(z)$ to be a polynomial of degree at most n.

6   Conclusions

The space of all time–dependent finite energy solutions to (1.3) is infinite–dimensional. Restricting to static solutions singles out finite–dimensional families (2.2). In this paper we have shown that a geodesic motion on the moduli space of static solutions approximates the non–relativistic dynamics of Ward solitons. Some of these approximate solutions have been related to exact uniton solutions of (1.3).

To construct finite–dimensional families of exact time–dependent solutions to (1.3), the finiteness of energy must be supplemented by other conditions. Ward [13] has shown that it is

sufficient to assume that the Higgs field Φ tends to 0, and the solution ψ of the associated linear problem (1.2) tends to $\mathbf{1}$ at spatial infinity of each spacelike plane. These conditions hold for all Yang–Mills–Higgs fields which arise from holomorphic vector bundles over the compactified twistor space. It would be interesting to understand how these more general finite energy solutions give rise to a motion on the moduli space of rational maps.

One expects that the integration of the equations of motion associated to (1.6, 1.7) could perhaps be made explicit because of the integrability of (1.1). The conservation of the energy (1.4), and the y–component of the momentum in field theory will yield two candidates for conserved quantities on the moduli space of static solutions, but they are not sufficient to ensure the solvability. We have already demonstrated that the kinetic energy gives rise to a conserved mechanical energy (2.5). The analogous procedure applied to the y–component of the momentum with a density

$$P_y = \mathrm{Tr}(J^{-1}J_y J^{-1}J_t)$$

gives, using (2.2),

$$P = \int \mathrm{Tr}(J^{-1}J_y J^{-1}J_t)\mathrm{d}x\mathrm{d}y = \int \frac{8\,\mathrm{Im}(f_z\dot{\overline{f}})}{(1+|f|^2)^2}\mathrm{d}x\mathrm{d}y = 8\pi\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Re}\,(a+b),$$

where a, b are given by (2.9), and the last equality follows from the application of Green's theorem along the lines which led to (2.10). The integrability of (1.3) guarantees the existence of an infinite sequence of 'hidden' conservation laws not related to the space–time symmetries and the Noether theorem. It remains to be seen whether these additional symmetries give rise to conservation laws on the moduli space which are sufficient to guarantee integrability in the sense of the Arnold–Liouville theorem.

The WZW term in the Lagrangian generates a magnetic field on the space of all fields; however, we showed that this vanishes on the moduli space. The resulting flat connection (2.7) could still be interesting, because the moduli space of based rational maps is not simply connnected. If non–trivial, it would imply that the the quantization of Ward solitons in the low-energy limit differs from the quantization of standard sigma model lumps.

Acknowledgements

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